Revision Notes

Class - 11 Physics

Chapter 4 - Motion in a Plane

1. SCALARS AND VECTORS

Some quantities can be described by a single number. For example, mass, time, distance and speed can be described using a single number. These are called scalar quantities.

To express someone how to get to a location from some other location, one piece of information is not enough. To describe this fully, both distance and displacement are required.

Quantities which require both magnitude and direction to describe a situation fully are known as vectors. For example, displacement and velocity are vectors.

The vectors are denoted by putting an arrow over the symbols representing them.

For example, AB vector can be represented by $\overrightarrow{AB}$.

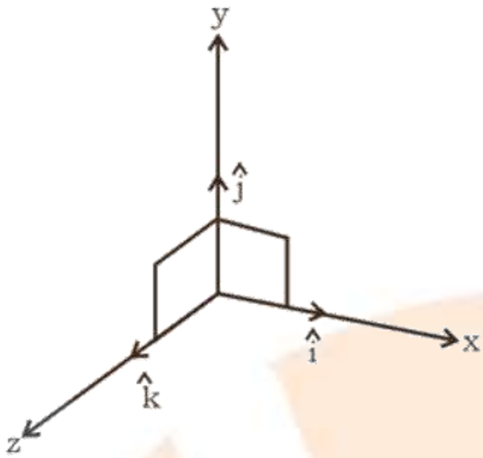
1.1 Unit vector

A unit vector has a magnitude of one and hence, it actually gives just the direction of the vector.

A unit vector can be determined by dividing the original vector by its magnitude

$$\Rightarrow \hat{a} = \frac{\vec{a}}{|\vec{a}|}$$

Unit vectors along different co–ordinate axis are as shown below:



1.2 Addition, subtraction and scalar multiplication of vectors

Consider two vectors as follows:

$$\vec{r}_1 = a_1\hat{i} + b_1\hat{j}$$

$$\vec{r}_2 = a_2\hat{i} + b_2\hat{j}$$

Then,

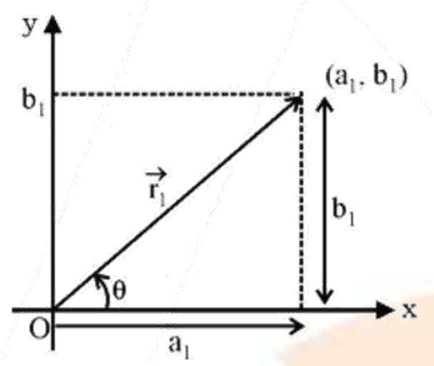
$$\vec{r}_1 + \vec{r}_2 = (a_1 + a_2)\hat{i} + (b_1 + b_2)\hat{j}$$

$$\vec{r}_1 - \vec{r}_2 = (a_1 - a_2)\hat{i} + (b_1 - b_2)\hat{j}$$

Multiplication of a vector by scalar quantity:

$$c\vec{r}_1 = c\left(a_1\hat{i} + b_1\hat{j}\right) = ca_1\hat{i} + cb_1\hat{j}$$

Representation of $\vec{r}_1$ on the co–ordinate axis:



Magnitude and direction of $\vec{r}_1$:

Magnitude of $\vec{r}_1 \left(\left|\vec{r}_1\right|\right) = \sqrt{a_1^{\ 2} + b_1^{\ 2}}$

Direction of $\vec{r}_1$ is given by

$$\tan\theta = \frac{b_1}{a_1} = \frac{\text{component } y-\text{axis}}{\text{component along } x-\text{axis}}$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{b_1}{a_1}\right)$$

1.3 Parallel vectors

Two vectors are parallel if and only if they have the same direction. When any vector is multiplied by a scalar, a vector parallel to the original vector is obtained.

If $b = ka$, then b and a are parallel vectors. Generally, to find if two vectors are parallel or not, we should find their unit vectors.

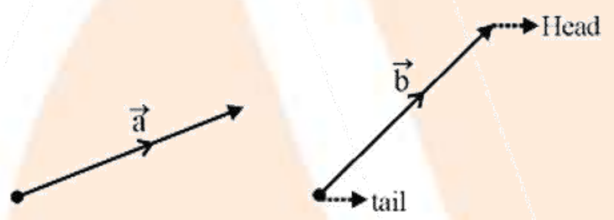
1.4 Equality of vectors

Two vectors (representing two values of the same physical quantity) are referred to as equal if their corresponding magnitudes and directions are the same.

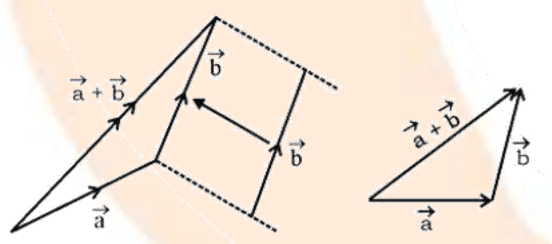
For example, $(3i+4j)m$ and $(3i+4j)\frac{m}{s}$ cannot be compared as they represent two different physical quantities.

1.5 Addition of vectors

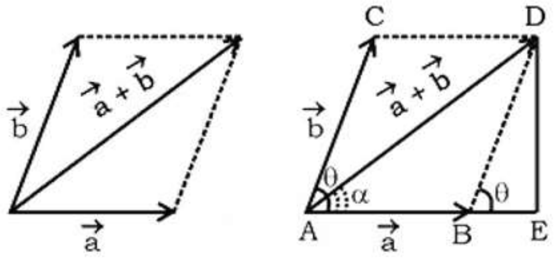
When two or more vectors are added, the answer is referred to as the resultant. The resultant of two vectors is equivalent to the first vector followed immediately by the second vector.



To determine the resultant of vectors a and b, the tail of vector b should be joined to the head of vector a. The resultant a+b is nothing but the direct vector from the tail of vector a to the head of vector b as shown below.



This is known as triangle rule of vector addition. Another way to obtain the resultant vector is parallelogram rule of vector addition. Here, we draw vectors $\vec{a}$ and $\vec{b}$, with both the tails coinciding. Taking these two as adjacent sides of a parallelogram, we complete the parallelogram. Now, the diagonal through the common tails gives the sum of two vectors.



Finding the magnitude of $\vec{a}+\vec{b}$ and its direction:

$$|AD|^2 = AE^2 + ED^2$$

Here,

$$AE = |\vec{a}| + |b\cos\theta|$$

$$ED = b\sin\theta$$

$$\Rightarrow AD^2 = a^2 + b^2\cos^2\theta + 2ab\cos\theta + b^2\sin^2\theta$$

$$\Rightarrow AD^2 = a^2 + b^2 + 2ab\cos\theta$$

$$\Rightarrow AD = \sqrt{a^2 + b^2 + 2ab\cos\theta}$$

where,

θ is the angle contained between $\vec{a}$ and $\vec{b}$;

Also,

$$\tan\alpha = \frac{ED}{AE} = \frac{b\sin\theta}{a + b\cos\theta}$$

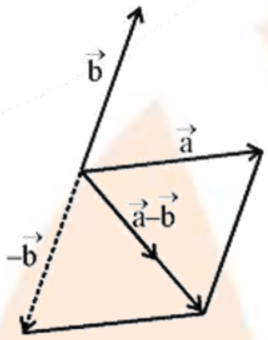
where,

α is the angle which the resultant makes with the positive x-axis.

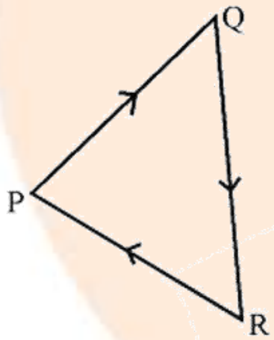
Subtraction of vectors

Let $\vec{a}$ and $\vec{b}$ be two vectors. We define $\vec{a}-\vec{b}$ as the sum of vectors $\vec{a}$ and the vector $(-\vec{b})$.

$\Rightarrow \vec{a}-\vec{b}=\vec{a}+(-\vec{b})$



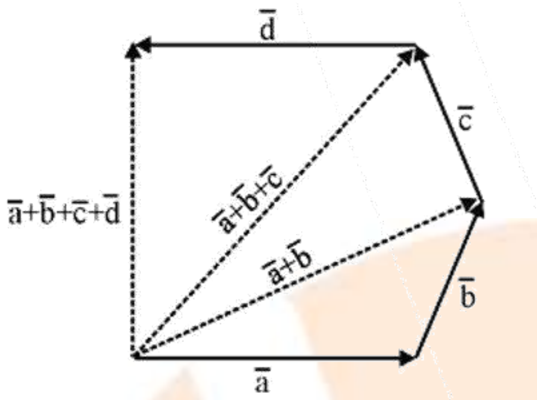
Zero vector

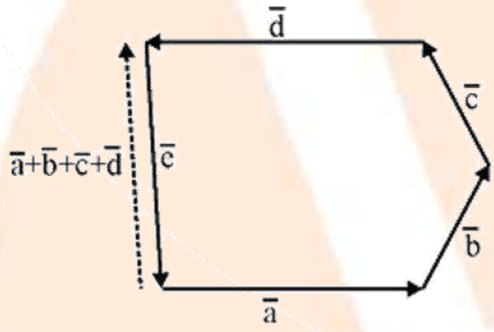


In the given triangle, $\overrightarrow{PQ}+\overrightarrow{QR}+\overrightarrow{PR}$ should be equal to zero as the overall journey results in a return to the starting point.

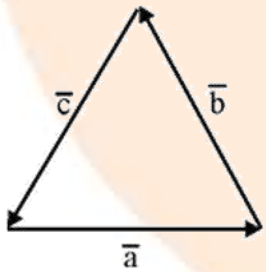
$\Rightarrow \overrightarrow{PQ}+\overrightarrow{QR}+\overrightarrow{PR}=0$

Here are some other resultants:



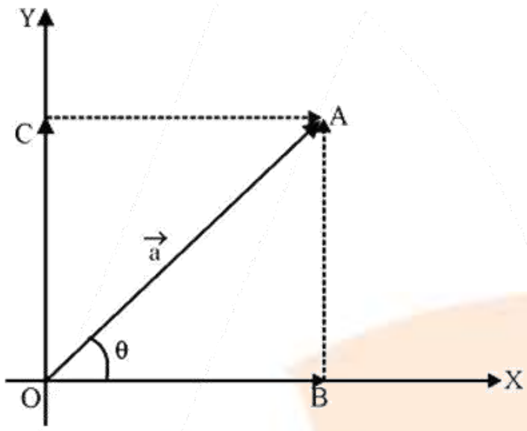


Here, $\vec{a}+\vec{b}+\vec{c}+\vec{d}+\vec{e}=0 \Rightarrow \vec{e}=-(\vec{a}+\vec{b}+\vec{c}+\vec{d})$



Here, $\vec{a}+\vec{b}+\vec{c}=0$

Resolution of vectors



Consider the given diagram above.

Here,

$\overrightarrow{OA} = \vec{a}$

By vector addition rule,

$\overrightarrow{OA} = \overrightarrow{OB} + \overrightarrow{OC}$

$\left|\overrightarrow{OB}\right| = a\cos\theta$

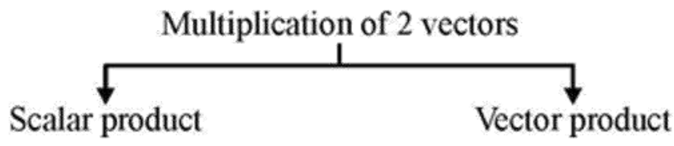
$\left|\overrightarrow{OC}\right| = a\sin\theta$

If $\hat{i}$ and $\hat{j}$ denote vectors of unit magnitude along OX and along OY respectively, we get

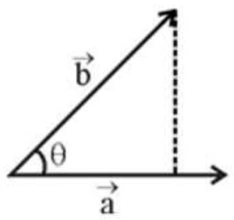
$\overrightarrow{OB} = a\cos\theta\,\hat{i}$

$\overrightarrow{OC} = a\sin\theta\,\hat{j}$

$\Rightarrow \vec{a} = (a\cos\theta)\hat{i} + (a\sin\theta)\hat{j}$



1.6 Dot product or scalar product of two vectors



Dot product of vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

If $\theta = 0;$

$$\Rightarrow \vec{a}.\vec{b} = |\vec{a}||\vec{b}|$$

If $\theta = 90^{\circ};$

$$\Rightarrow \vec{a}.\vec{b} = |\vec{a}||\vec{b}|90^{\circ} = 0$$

Dot product of unit vectors are given by

$$\hat{i}\cdot\hat{i} = |\hat{i}||\hat{i}|\cos 0^{\circ} = i^2 \times 1 = 1$$

Similarly,

$$\hat{j}\cdot\hat{j} = 1;$$

$$\hat{k}\cdot\hat{k} = 1$$

Now,

$$\hat{i}\cdot\hat{j} = |\hat{i}||\hat{j}|\cos 90^{\circ} = 1 \times 1 \times 0 = 0$$

Similarly,

$$\hat{j}\cdot\hat{k} = 0\ ;$$

$\hat{k} \cdot \hat{i} = 0$

Dot products are commutative and distributive:

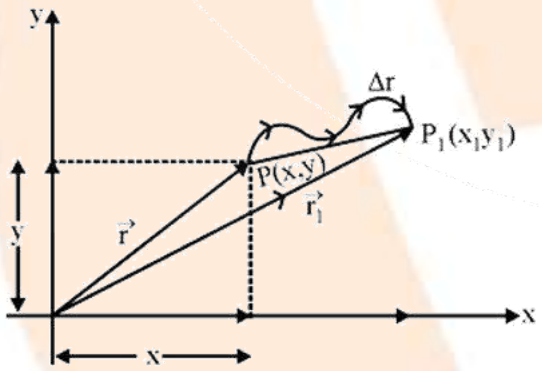
$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

$\vec{a} \cdot \left(\vec{b} + \vec{c}\right) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

2. MOTION IN 2D (PLANE)

2.1 Position vector and Displacement

The position vector $\vec{r}$ of a particle P, located in a plane with reference to the origin of on xy–coordinate system is given by $\vec{r} = x\hat{i} + y\hat{j}$, as shown below.



Now, if the particle moves along the path as shown to a new position P_1 with the position vector $\vec{r}_1$;

$\vec{r}_1 = x_1\hat{i} + y_1\hat{j}$

Change in position of the particle is nothing but its displacement given by,

$\Delta\vec{r} = \vec{r}_1 - \vec{r} = \left(x_1\hat{i} + y_1\hat{j}\right) - \left(x\hat{i} + y\hat{j}\right)$

$\Rightarrow \Delta\vec{r} = (x_1 - x)\hat{i} + (y_1 - y)\hat{j}$

$\Delta\vec{r} == \Delta x\hat{i} + \Delta y\hat{j}$

From the figure, it can also be seen that

$\vec{r} + \Delta\vec{r} = \vec{r}_1$ or $\Delta\vec{r} = \vec{r}_1 - \vec{r}$, which is nothing but the triangle law of vector addition.

2.2 Average velocity

Average velocity is given by,

$$\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\Delta x\hat{i} + \Delta y\hat{j}}{\Delta t}$$

$$\Rightarrow \vec{v}_{avg} = v_x\hat{i} + v_y\hat{j}$$

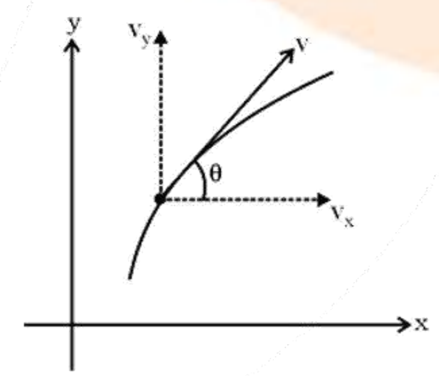
Note: Direction of the average velocity is the same as that of $\Delta\vec{r}$.

2.3 Instantaneous velocity

Instantaneous velocity is given by,

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{d\vec{r}}{dt}$$

$$\Rightarrow \vec{v} = v_x\hat{i} + v_y\hat{j}$$



Here,

$$v_x = \frac{dx}{dt} \text{ and } v_y = \frac{dy}{dt}$$

$$\Rightarrow |\vec{v}| = \sqrt{{v_x}^2 + {v_y}^2}$$

Also,

$$\tan\theta = \frac{v_y}{v_x}$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$

Note: The direction of instantaneous velocity at any point on the path of an object is the tangent to the path at that point and is in the direction of motion.

2.4 Average acceleration

$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right)$ Average acceleration is given by,

$$\vec{a}_{avg} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\Delta v_x}{\Delta t}\hat{i} + \frac{\Delta v_y}{\Delta t}\hat{j}$$

$$\Rightarrow \vec{a}_{avg} = a_x\hat{i} + a_y\hat{j}$$

2.5 Instantaneous acceleration

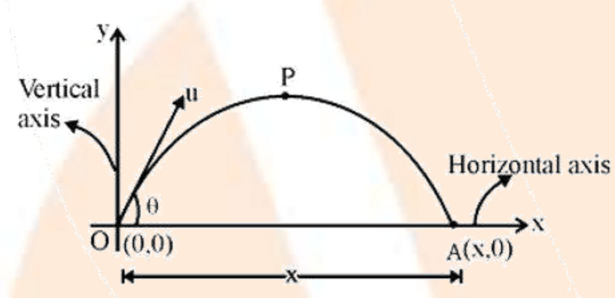
Instantaneous acceleration is given by,

$$\vec{a} = \frac{dv}{dt} = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j}$$

$$\Rightarrow \vec{a} = a_x\hat{i} + a_y\hat{j}$$

3. PROJECTILE MOTION

When a particle is projected obliquely close to the surface of the earth, it moves simultaneously in horizontal and vertical directions. Motion of such a particle is referred to as projectile motion.



Here, a particle is projected at an angle with an initial velocity 'u'.

Considering the projectile motion given in the diagram above, let us calculate the following:

(a) time taken to reach A from O

(b) horizontal distance covered (OA)

(c) maximum height reached during the motion

(d) velocity at any time 't' during the motion

Horizontal axis	Vertical axis
$u_x = u\cos\theta$ $a_x = 0$ (In the absence of any external force, a_x would be assumed to be zero).	$u_y = u\sin\theta$ $a_y = -g$ $s_y = u_y t + \frac{1}{2}a_y t^2$

	$\Rightarrow 0-0=u\sin\theta t-\frac{1}{2gt^{2}}$ $\Rightarrow \boxed{T=\frac{2u\sin\theta}{g}=\frac{2u_y}{g}}$
$s_x=u_xt+\frac{1}{2a_xt^{2}}$ $\Rightarrow x-0=u\cos\theta t$ $\Rightarrow x=u\cos\theta\times 2u\sqrt{g}$ $\Rightarrow x=\frac{2u^{2}\cos\theta\sin\theta}{g}$ $\Rightarrow \boxed{R=\frac{u^{2}\sin 2\theta}{g}}$ $(\because 2\cos\theta\sin\theta=\sin 2\theta)$ Horizontal distance covered is known as **Range (R).**	$v_y=u_y+a_yt$ $\boxed{v_y=u\sin\theta-gt}$ It depends on time ‘t’. It is not constant. It’s magnitude first decreases; becomes zero and then increases.
$v_x=u_x+a_xt$ $\boxed{v_x=u\cos\theta}$ It is independent of t. It is constant.	**Maximum height** obtained by the particle Method 1: Using time of ascent; $s_y=u_yt_1+\frac{1}{2}ayt_1^{2}$
Time of ascent and time of descent: At top most point, $v_y=0$	$\Rightarrow \boxed{H=\frac{u^{2}\sin^{2}\theta}{2g}}$

$v_y = u_y + a_y t$ $\Rightarrow 0 = u\sin\theta - gt$ $\Rightarrow t_1 = \frac{u\sin\theta}{g}$ $\Rightarrow t_2 = T - t_1 = \frac{u\sin\theta}{g}$ $\Rightarrow \boxed{t_1 = t_2 = \frac{T}{2} = \frac{u\sin\theta}{g}}$	Method 2: Using third equation of motion $v_y^{\ 2} - u_y^{\ 2} = 2a_y s_y$ $\Rightarrow 0 - u^2\sin^2\theta = -2gs_y$ $\Rightarrow \boxed{H = \frac{u^2\sin^2\theta}{2g}}$

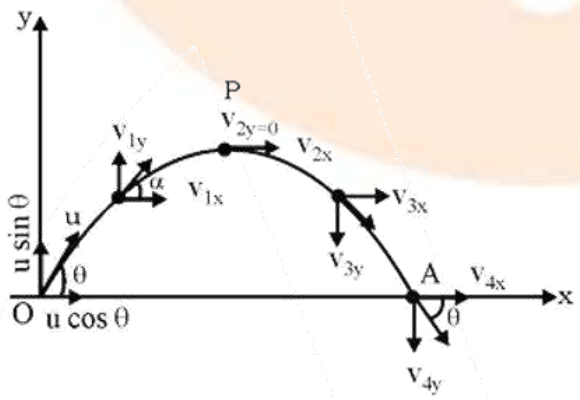
Maximum Range

$R = \frac{u^2\sin 2\theta}{g}$ and $R_{max} = \frac{u^2}{g}$

Range is maximum when $\sin 2\theta$ is maximum;

$\Rightarrow \max(\sin 2\theta) = 1$ or $\theta = 45^{\circ}$

3.1 Analysis of velocity in case of a projectile



From the above equations;

i) $v_{1x} = v_{2x} = v_{3x} = v_{4x} = u_x = u\cos\theta$

which suggests that the velocity along x axis remains constant.

[as there is no external force acting along that direction]

ii)

a) magnitude of velocity along y axis first decreases and then it increases after the top most point.

b) At the top most point, magnitude of velocity is zero.

c) Direction of velocity is in the upward direction while ascending and is in the downward direction while descending.

d) Magnitude of velocity at A is the same as magnitude of velocity at O; but the directions are opposite.

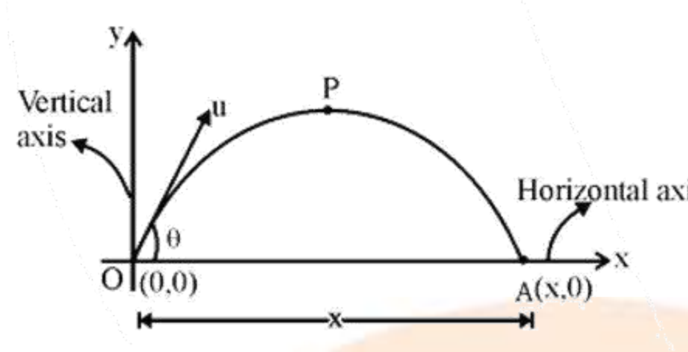
e) Angle which the net velocity makes with the horizontal can be evaluated by,

$$\tan\alpha = \frac{v_y}{v_x} = \frac{\text{velocity along y axis}}{\text{velocity along x axis}}$$

f) Net velocity is always along the tangent.

3.2 Equation of trajectory

Trajectory refers to the path traced by the body. To determine the trajectory, we should find the relation between y and x by eliminating time.

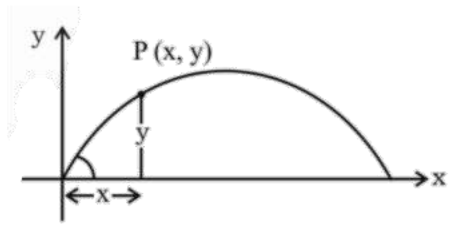


Horizontal Motion	Vertical Motion
$u_x = u\cos\theta$ $a_x = 0$ $s_x = u\cos\theta\, t = x$ $\Rightarrow t = \dfrac{x}{u\cos\theta}$	$u_y = u\sin\theta$ $a_y = -g$ $s_y = u_y t + \dfrac{1}{2}a_y t^2$ $\Rightarrow y = u\sin\theta\left(\dfrac{x}{u\cos\theta}\right) - \dfrac{1}{2}g\dfrac{x^2}{u^2\cos^2\theta}$

$$\boxed{y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta} \Rightarrow y = bx - ax^2}$$

(i) This is the equation of a parabola.

(ii) Because the coefficient of x_2 is negative, it is an inverted parabola.



Path of the projectile is a parabola.

$$R = \frac{2u^2 \sin\theta \cos\theta}{g}$$

$$\Rightarrow \frac{2u^2}{g} = \frac{R}{\sin\theta\cos\theta}$$

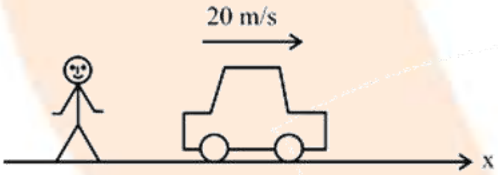
Substituting this value in the above equation, we have,

$$\Rightarrow \boxed{y = x\tan\theta\left[1 - \frac{x}{R}\right]}$$

4. RELATIVE MOTION

Relativity is a very common term. In physics, we use relativity very oftenly.

For example, consider a moving car and yourself (observer) as shown below.



Case I: If you are observing a car moving on a straight road, then you say that the velocity of car is 20m/s; which means that velocity of car relative to you is 20m/s; or, velocity of car relative to the ground is 20m/s (as you are standing on the ground.

Case II: If you go inside this car and observe, you would find that the car is at rest while the road is moving backwards. Then, you would say, the velocity of car relative to the car is 0m/s.

Mathematically, velocity of B relative to A is represented as

$$\vec{v}_{BA} = \vec{v}_B - \vec{v}_A$$

This, being a vector quantity, direction is very important.

$$\therefore \vec{v}_{BA} \neq \vec{v}_{AB}$$

5. RIVER–BOAT PROBLEMS

In river–boat problems, we come across the following three terms:

$\vec{v}_r =$ absolute velocity of river.

$\vec{v}_{br} =$ velocity of boatman with respect to river or velocity of boatman in still water, and

$\vec{v}_b =$ absolute velocity of boatman.

Clearly, it is important to note that $\vec{v}_{br}$ is the velocity of boatman with which he steers and $\vec{v}_b$ is the actual velocity of boatman relative to ground. Further,

$$\vec{v}_b = \vec{v}_{br} + \vec{v}_r$$

Now, let us derive a few standard results and their special cases.

A boatman starts from point A on one bank of a river with velocity $\vec{v}_{br}$ in the direction shown in figure. River is flowing along positive x-direction with velocity $\vec{v}_r$. Width of the river is ‘d’. Then,

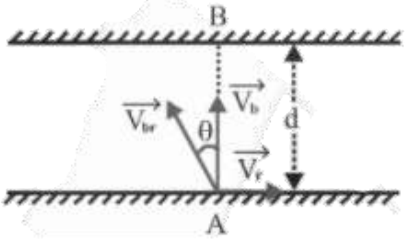
$$\vec{v}_b = \vec{v}_{br} + \vec{v}_r$$

Therefore,

$$v_{bx} = v_{rx} + v_{brx} = v_r - v_{br}\sin\theta$$

And

$$v_{by} = v_{by} + v_{bry} = 0 + v_{br}\cos\theta = v_{br}\cos\theta$$



Now, the time taken by the boatman to cross the river is given by,

$$t=\frac{d}{v_{by}}=\frac{d}{v_{br}\cos\theta}$$

$$\Rightarrow t=\frac{d}{v_{br}\cos\theta} \quad \ldots (1)$$

Further, displacement along the x-axis when he reaches on the other bank (also called as drift) is given by,

$$x=v_{bx}t=\left(v_r-v_{br}\sin\theta\right)\frac{d}{v_{br}\cos\theta}$$

$$\Rightarrow x=\left(v_r-v_{br}\sin\theta\right)\frac{d}{v_{br}\cos\theta} \ldots (2)$$

Condition when the boatman crosses the river in shortest interval of time:

From (1), it can be seen that time (t) will be minimum when $\theta = 0°$ i.e., the boatman should steer his boat perpendicular to the river current.

Condition when the boat wants to reach point B, i.e., at a point just opposite from where he started (shortest distance):

In this case, the drift (x) should be zero.

$$\Rightarrow x=0$$

$$\Rightarrow \left(v_r-v_{br}\sin\theta\right)\frac{d}{v_{br}\cos\theta}=0$$

$$\Rightarrow v_r = v_{br} \sin\theta$$

$$\Rightarrow \sin\theta = \frac{v_r}{v_{br}}$$

$$\Rightarrow \theta = \sin^{-1}\left(\frac{v_r}{v_{br}}\right)$$

Clearly, to reach point B, the boatman should row at an angle $\theta = \sin^{-1}\left(\frac{v_r}{v_{br}}\right)$ upstream from AB.

Now, time is given by,

$$t = \frac{d}{v_b} = \frac{d}{\sqrt{{v_{br}}^2 - {v_r}^2}}$$

As $\sin\theta$ is not greater than 1 and when $v_r \geq v_{br}$, then the boatman can never reach at point B.

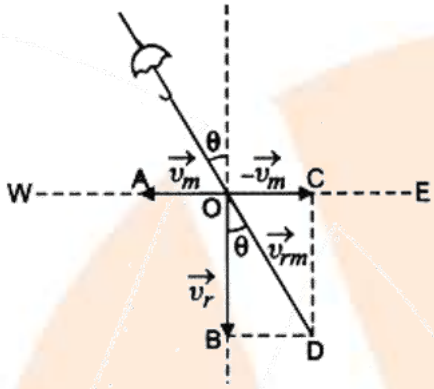
Because, when $v_r = v_{br}$, $\sin\theta = 1$ or $\sin\theta = 90°$ and it is just impossible to reach at B if $\theta = 90°$.

Similarly, when $v_r > v_{br}$, $\sin\theta > 1$, i.e., no such angle exists. Practically, it can be realized in this manner that it is not possible to reach at B if the river velocity (v_r) is too high.

6. RELATIVE VELOCITY OF RAIN WITH RESPECT TO A MOVING MAN

Consider a man walking west with velocity $\vec{v}_m$, represented by OA. Let the rain be falling vertically downwards with velocity $\vec{v}_r$, represented by OB, as shown in the following figure. To find the relative velocity of rain with respect to man (i.e., $\vec{v}_{rm}$) assume the man to be at rest by imposing a velocity $-\vec{v}_m$ on the man and apply this velocity on rain also. Now, the relative velocity of rain with respect to man would be the resultant velocity of $\vec{v}_r(=\overrightarrow{OB})$ and $-\vec{v}_m\left(=\overrightarrow{OC}\right)$, which would be represented by the diagonal OD of rectangle OBDC.

$$\Rightarrow v_{rm} = \sqrt{{v_r}^2 + {v_m}^2 + 2v_r v_m \cos 90^\circ} = \sqrt{{v_r}^2 + {v_m}^2}$$



If θ is the angle which v_{rm} makes with the vertical direction, then

$$\tan\theta = \frac{BD}{OB} = \frac{v_m}{v_r}$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{v_m}{v_r}\right)$$

Here, angle θ is from the vertical towards west and is written as θ, west of vertical.

Note: In the above problem, if the man wants to protect himself from the rain, he should hold his umbrella in the direction of relative velocity of rain with respect to man. i.e., the umbrella must be held making an angle $\left(\theta = \tan^{-1}\left(\frac{v_m}{v_r}\right)\right)$, west of the vertical.